Why is everything multiplied by nine when the digits are added the result is nine?

4

Why is it that whenever you multiply any number by nine the digits of the result always add up to nine? Example:

*9 * 7 = 63; // 6 + 3 = 9*
*9 * 35 = 315; // 3 + 1 + 5 = 9*

elementary-number-theory

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edited Oct 4 '12 at 12:19
rschwieb
79k 10 66 174

asked Oct 4 '12 at 12:07
Yehuda
21 2

$11 \cdot 9 = 99, 9+9=18, 1111*9=9999, 9+9+9+9=36?$ 11·9=99,9+9=18,1111∗9=9999,9+9+9+9=36? It should be mutiple of 9? – lab bhattacharjee Oct 4 '12 at 12:10

@labbhattacharjee The poster probably meant that you continue adding digits until you get one digit. 1+8=9 and 3+6=9 – rschwieb Oct 4 '12 at 12:17

2 Much more usefully, if the result of this process *does* produce 99, then your original number is a multiple of 99.– Sean Eberhard Oct 4 '12 at 12:23

2 In all the proofs the crucial fact is that $10 \equiv 1 \ \ (9)$ 10≡1 (9). – PAD Oct 4 '12 at 13:36

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4 Answers

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5

There is a well-known divisibility test that a number is divisble by 99 if and only if 99 divides the sum of its digits.

By multiplying a number with 99, you are making it a multiple of 99, and hence the sum of its digits must be divisible by 99. Since you can iterate the digit adding process until you get down to a single digit number, the result must be divisible by 99.

The only two single digit numbers divisible by 99 are 00 and 99, and you're never going to get 00 (well, unless your original number was 0 :)).

Something similar works for 33, but it is not as nice. If you multiply a number by 3 and then iterate the digit adding process until you have one digit, then you will wind up with 3, 63,6 or 99. The reason is the same: a number is divisble by 3 if and only if the sum of its digits is. However this time when you get down to single digits, there is more than one alternative that you can arrive at.

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edited Oct 4 '12 at 12:30

answered Oct 4 '12 at 12:19
rschwieb
79k 10 66 174

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4

The digit sum of a positive integer is always congruent to itself modulo 99. Since the digit sum is always less than the original number, and your original number is a mutliple of 99, repeating the process boils it down to 99.

Why? Write your integer as
$a_0 + 10a_1 + 10^2a_2 + \cdots 10^n a_n \equiv 0 \pmod{9}, \quad n > 0$ a0+10a1+102a2+⋯10nan≡0(mod9),n>0

For every $0 \le k \le n, \quad 10^k a_k \equiv a_k \pmod{9}$ 0≤k≤n,10kak≡ak(mod9) therefore:

$$a_0 + 10a_1 + 10^2a_2 + \cdots 10^n a_n \equiv a_0 + a_1 + a_2 + \cdots a_n \equiv 0 \pmod{9}$$
a0+10a1+102a2+⋯10nan≡a0+a1+a2+⋯an≡0(mod9)

However,

$$a_0 + 10a_1 + 10^2a_2 + \cdots 10^n a_n > a_0 + a_1 + a_2 + \cdots a_n, \quad (n > 0)$$
a0+10a1+102a2+⋯10nan>a0+a1+a2+⋯an,(n>0)

Thus, repeated descent brings us to smallest positive integer congruent to $0 \pmod{9}$ 0(mod9), which is 99.

share cite improve this answer edited Oct 4 '12 at 12:46 answered Oct 4 '12 at 12:19
user39572

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1

Let $p(x)$ p(x) be a polynomial, $r \in \mathbb{R}$ r∈R. Define $q(x) := p(x) - p(r)$ q(x):=p(x)−p(r). Since $q(r) = 0$ q(r)=0, we can write $q(x) = s(x)(x - r)$ q(x)=s(x)(x−r) for some polynomial $s(x)$ s(x). This yields

$$p(x) - p(r) = s(x)(x - r)$$
$$\Leftrightarrow \quad p(x)/(x - r) = s(x) + p(r)/(x - r)$$
p(x)−p(r)=s(x)(x−r)⇔p(x)/(x−r)=s(x)+p(r)/(x−r)

Now given any number $n \in \mathbb{N}$ n∈N, there is a polynomial $p_n(x)$ pn(x), such that $p_n(10) = n$ pn(10)=n. Hence

$$n/9 = p_n(10)/(10 - 1) = s_n(10) + p_n(1)/9$$
n/9=pn(10)/(10−1)=sn(10)+pn(1)/9

Hence 9 divides $n = p(10)$ n=p(10) if and only if 9 divides $p_n(1)$pn(1), which is the sum of all digits of nn.

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roman
636 3 13

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0

Start with

$$9 \cdot n = 9\sum_{k=0}^{\infty} a_k 10^k = 9(a_0 10^0 + a_1 10^1 + a_2 10^2 + \cdots) = 9a_0 10^0 + 9a_1 10^1 + 9a_2 10^2 + \cdots$$

$$9\cdot n=9\sum_{k=0}^{\infty}a_k10^k=9(a_0 10^0+a_1 10^1+a_2 10^2+\cdots)=9a_0 10^0+9a_1 10^1+9a_2 10^2+\cdots$$

Then the decimal representation of $9c$ is $[c-1, 10-c]$ $[c-1,10-c]$, if $c>0$ $c>0$.

So $9a_k10^k=(a_k-1)10^{k+1}+(10-a_k)10^k$ $9a_k10^k=(a_k-1)10^{k+1}+(10-a_k)10^k$, if $a_k>0$ $a_k>0$. Adding up all digits of the prodcut will give

$$\sum(a_k-1)+(10-a_k)=\sum a_k-1+10-a_k=\sum 9.$$

$$\sum(a_k-1)+(10-a_k)=\sum a_k-1+10-a_k=\sum 9.$$

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answered Oct 4 '12 at 12:34